

# EQUALITY OF SYMMETRIZED TENSORS AND THE COORDINATE RING OF THE FLAG VARIETY

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## 1. INTRODUCTION

The goal of this note is to give a transparent proof of a result of da Cruz and Dias da Silva on the equality of symmetrized decomposable tensors. This will be done by explaining that their result follows from the fact that the coordinate ring of a flag variety is a unique factorization domain.

Let  $\lambda$  be a partition of a positive integer  $n$  and let  $\chi^\lambda$  be the irreducible character of the symmetric group  $\mathfrak{S}_n$  corresponding to  $\lambda$ . There is a right action of  $\mathfrak{S}_n$  on  $V^{\otimes n}$ , where  $V$  is a finite-dimensional complex vector space, by permuting tensor positions. Let  $T_\lambda$  be the endomorphism of  $V^{\otimes n}$  given by

$$(v_1 \otimes \cdots \otimes v_n)T_\lambda = \frac{\chi^\lambda(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

The question of when there is an equality,

$$(1) \quad (v_1 \otimes \cdots \otimes v_n)T_\lambda = (u_1 \otimes \cdots \otimes u_n)T_\lambda$$

was studied by Merris [12], and Chollet and Marcus [2, 11] in the mid-1970s. Since then, many papers were written on this question and partial results were given by a variety of authors over the years (for example, [3, 6, 7, 8, 9]). However, the question was fully resolved only recently in two papers by da Cruz and Dias da Silva [4, 5].

A tableau of shape  $\lambda$  will be a filling of the numbers  $1, 2, \dots, n$  into the boxes of the Young diagram of  $\lambda$ . A tableaux is said to be standard if the numbers in both its rows and columns increase.

**Theorem** (da Cruz–Dias da Silva). *There is an equality of symmetrized decomposable tensors as in (1) if and only if the following conditions are satisfied:*

- (1) *Every tableau of shape  $\lambda$  whose columns index linearly independent subsets of  $(v_1, \dots, v_n)$  also index linearly independent subsets of  $(u_1, \dots, u_n)$ .*
- (2) *If the columns of a tableau of shape  $\lambda$  index independent subsets of  $(v_1, \dots, v_n)$ , and  $C_j$  denotes the numbers in column  $j$  of this tableaux, then there is some permutation  $\sigma$  such that for all  $j$ , the span of those  $\{v_i : i \in C_j\}$  is equal to the span  $\{u_i : i \in C_{\sigma(j)}\}$ .*
- (3) *In the situation above, the product the determinants relating  $\{v_i : i \in C_j\}$  to  $\{u_i : i \in C_{\sigma(j)}\}$  is 1.*

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The idea of the proof, and the structure of the paper, is as follows. We first reduce to the case of applying Young symmetrizers to the decomposable tensors. Then we use the fact that applying a Young symmetrizer to a tensor can be viewed as multiplication in the coordinate ring of a flag variety. The proof of the theorem follows by interpreting it as a statement about unique factorization in this ring.

The proof is straightforward, once a person becomes familiar with the various guises of the representation theory of the general linear group. Perhaps the myriad of ways of understanding these representations explains why the result withstood understanding for so long. Indeed, the description of the coordinate ring of a flag variety used here goes back to Deruyts, and a modern treatment of it was known to, among others, Towber [13] in the 1970s. That this coordinate ring is a unique factorization domain is what makes the theorem non-trivial. It is a vestige of the first fundamental theorem of invariant theory, as explained in [10, Chapter 9].

A certain amount of familiarity the representation theory of the general linear group will be assumed. We point the reader to Fulton's book [10] for a beautiful synthesis of all the relevant ideas, giving explicit pointers to particular results as we use them.

In point of notation, we will write  $v^\otimes$  for the tensor product  $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ .

All the results in this paper hold not just over  $\mathbf{C}$ , but over an arbitrary field of characteristic zero.

## 2. REDUCTION

Let  $T$  be a tableau of shape  $\lambda$ ,  $a_T$  its row symmetrizer, and  $b_T$  its column antisymmetrizer. These are given by

$$\sum_{\sigma \in \text{Row}(T)} \sigma, \quad \sum_{\sigma \in \text{Col}(T)} \text{sign}(\sigma) \sigma,$$

respectively. Here the sums are over the row and column groups of  $T$ , which are the subgroups of  $\mathfrak{S}_n$  that stabilize each row and each column of  $T$ , respectively. For example, using cycle notation for permutations in  $\mathfrak{S}_n$ , if

$$T = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 1 & 5 & \\ \hline \end{array}$$

then  $b_T = (1 - (12))(1 - (35))$  while

$$a_T = (1 + (23) + (24) + (34) + (234) + (243))(1 + (15)).$$

A product  $c_T := b_T a_T$  is called a *Young symmetrizer* and the right ideal in  $\mathbf{C}\mathfrak{S}_n$  generated by a Young symmetrizer is an irreducible  $\mathbf{C}\mathfrak{S}_n$ -module with character  $\chi^\lambda$  [10, Chapter 7] while the image of  $b_T a_T$  on  $V^{\otimes n}$  is zero, or irreducible for the diagonal action of the general linear group with highest weight  $\lambda$  [10, Proposition 8.1]. It is clear that  $v^\otimes b_T$  is not zero if and only if the sets of vectors indexed by the columns of the tableau  $T$  are linearly independent.

**Proposition 1.** *There is an equality  $v^\otimes T_\lambda = u^\otimes T_\lambda$  if and only if for all tableau  $T$  of shape  $\lambda$  there is an equality  $v^\otimes c_T = u^\otimes c_T$ .*

*Proof.* Recall that  $T_\lambda$  is the projector of  $\mathbf{C}\mathfrak{S}_n$  to its  $\chi^\lambda$  isotypic component. Now,

$$v^\otimes T_\lambda = u^\otimes T_\lambda \implies v^\otimes T_\lambda c_T = u^\otimes T_\lambda c_T \implies v^\otimes c_T = u^\otimes c_T.$$

The first implication is trivial and the second follows from the fact that  $T_\lambda$  fixes any Young symmetrizer  $c_T$  in  $\mathbf{C}\mathfrak{S}_n$  if  $T$  has shape  $\lambda$ .

For the other implication, recall the fact that the sum  $\sum_T c_T$  over all tableaux of shape  $\lambda$  is a non-zero scalar multiple of  $T_\lambda^1$ . Hence, if  $v^\otimes c_T = u^\otimes c_T$  for all  $T$  of shape  $\lambda$  then,

$$\sum_T v^\otimes c_T = \sum_T u^\otimes c_T \implies v^\otimes T_\lambda = u^\otimes T_\lambda. \quad \square$$

### 3. FLAG VARIETIES AND THE SHAPE ALGEBRA

Denote the dimension of  $V$  by  $r$ . Let  $\mathcal{F}\ell_r$  be the complex variety of complete flags in  $V$ :

$$\mathcal{F}\ell_r = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_{r-1} \subset V : \dim V_i = i, i \leq r\}.$$

The coordinate ring of  $\mathcal{F}\ell_r$ , sometimes called the shape algebra [13], can be written as a quotient

$$A = A(\mathcal{F}\ell_r) := \text{Sym}(\bigwedge(\mathbf{C}^r)) / Q,$$

where  $\text{Sym}$  and  $\bigwedge$  denote the symmetric and exterior algebra functors, and  $Q$  is an ideal of quadratic relations whose precise definition is not needed (see [10, Proposition 9.1] for this). We can write this symmetric algebra of  $\bigwedge \mathbf{C}^r$  as,

$$\text{Sym}(\bigwedge(\mathbf{C}^r)) = \bigoplus_{a_1, a_2, \dots, a_r \geq 0} \text{Sym}^{a_r}(\bigwedge^r \mathbf{C}^r) \otimes \cdots \otimes \text{Sym}^{a_1}(\bigwedge^1 \mathbf{C}^r),$$

and the ideal  $Q$  is homogeneous with respect to the obvious  $\mathbb{N}^r$ -grading. The graded piece of the quotient  $A$  that is indexed by  $(a_1, a_2, \dots, a_r)$  can, instead, be indexed by the partition that has  $a_i$  columns of length  $i$ , for  $i = 1, 2, \dots, r$ . We write the graded piece of  $A$  corresponding to  $\lambda$  as  $A_\lambda$ . In this way we see that the coordinate ring of  $\mathcal{F}\ell_r$  can be written as a direct sum  $A = \bigoplus A_\lambda$ , the sum over partitions  $\lambda$  with at most  $r$  parts.

If we have a partition  $\lambda$  with  $a_i$  columns of length  $i$  for  $1 \leq i \leq r$ , then it is a fact that  $A_\lambda$  is the irreducible representation of  $\text{GL}(V)$  with highest weight  $\lambda$  [10, Theorem 8.2]. Further, for all tableau  $T$  of shape  $\lambda$  there is a unique isomorphism of  $\text{GL}_r(V)$  representations [10, Proposition 8.1],

$$A_\lambda \approx V^{\otimes n} c_T.$$

Let  $T$  denote the *column super standard* tableau of shape  $\lambda$ , whose entries are  $1, 2, \dots, n$  when read top-to-bottom, left-to-right. By the universal property of  $A_\lambda$  [10, Theorem 8.1] note that the composition of multiplication maps,

$$\begin{aligned} (\bigwedge^r \mathbf{C}^r)^{\otimes a_r} \otimes \cdots \otimes (\bigwedge^1 \mathbf{C}^r)^{\otimes a_1} &\rightarrow \text{Sym}^{a_r}(\bigwedge^r \mathbf{C}^r) \otimes \cdots \otimes \text{Sym}^{a_1}(\bigwedge^1 \mathbf{C}^r) \\ &\rightarrow A_\lambda \rightarrow V^{\otimes n} c_T, \end{aligned}$$

takes an element,

$$(v_1 \wedge \cdots \wedge v_r) \otimes \cdots \otimes (v_{(r-1)a_r} \wedge \cdots \wedge v_{ra_r}) \otimes \cdots \otimes v_{n-a_1} \otimes \cdots \otimes v_n,$$

to  $(v_1 \otimes v_2 \otimes \cdots \otimes v_n) c_T$ .

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<sup>1</sup>This follows from the fact  $\sum_T c_T$  is not zero in  $\mathbf{CS}_n$ , central, idempotent, and annihilates Young symmetrizers of different shapes.

#### 4. PROOFS

We can now, in one fell swoop, prove two results of Dias da Silva and Fonseca (unpublished), as well as the theorem of da Cruz and Dias da Silva.

**Theorem 2** (Dias da Silva–Fonseca). *The tensor  $v^{\otimes} c_T$  is not zero if and only if  $v^{\otimes} b_T$  is not zero if and only if the columns of  $T$  index linearly independent subsets of  $(v_1, \dots, v_n)$ .*

*Proof.* Let  $C_j$  denote the set of numbers in the  $j$ th column of  $T$ . Then  $v^{\otimes} c_T$  is not zero if and only if the product of the elements

$$\bigwedge_{i \in C_1} v_i, \bigwedge_{i \in C_2} v_i, \bigwedge_{i \in C_3} v_i, \dots$$

is not zero in  $A$ . However,  $A$  is an integral domain [10, Proposition 8.2] hence the product is not zero if and only if each of its factors is not zero. The latter is true if and only if each set of vectors  $\{v_i : i \in C_j\}$  is linearly independent.  $\square$

Combining this result with Proposition 1 we have given another proof of Gamas's theorem on the vanishing of symmetrized decomposable tensors (*cf.*, [1]).

**Corollary 3** (Gamas). *The symmetrized tensor  $v^{\otimes} T_\lambda$  is not zero if and only there is a tableau of shape  $\lambda$  whose columns index linearly independent sets.*

The following slightly stronger result was first due to Dias da Silva and Fonseca (unpublished).

**Corollary 4** (Dias da Silva–Fonseca). *The symmetrized tensor  $v^{\otimes} T_\lambda$  is not zero if and only there is a standard tableau of shape  $\lambda$  whose columns index linearly independent sets.*

Recall that a tableau is standard if the numbers in each row and column increase.

*Proof.* Suppose that  $v^{\otimes} c_T$  is not zero, but  $T$  is not a standard tableau. It is well known that there is an equality of the form  $c_T = \sum_j c_S x_S$ , where the sum is over standard tableaux of shape  $\lambda$  and  $x_S \in \mathbf{CS}_n$ . The result follows.  $\square$

We now give the promised proof of the theorem on equality of symmetrized decomposable tensors.

**Theorem 5** (da Cruz–Dias da Silva). *There is an equality of symmetrized decomposable tensors as in (1) if and only if the following conditions are satisfied:*

- (1) *Every tableau of shape  $\lambda$  whose columns index a linearly independent subsets of  $(v_1, \dots, v_n)$  also index linearly independent subsets of  $(u_1, \dots, u_n)$ .*
- (2) *If the columns of a tableau of shape  $\lambda$  index independent subsets of  $(v_1, \dots, v_n)$  and  $C_j$  denotes the numbers in column  $j$  of this tableaux, then there is some permutation  $\sigma$  such that for all  $j$ ,*

$$\bigwedge_{i \in C_j} v_i = c_j \bigwedge_{i \in C_{\sigma(j)}} u_i$$

*for some non-zero scalar  $c_j$ . That is, for all  $j$ , the span of those  $v_i$  indexed by  $C_j$  is equal to the span of those  $u_i$  indexed by  $C_{\sigma(j)}$ .*

- (3) *In the situation above, the product of all the  $c_j$ s is 1.*

*Proof.* After relabeling the vectors, it is sufficient to check that  $v^{\otimes} c_T = u^{\otimes} c_T$  for  $T$  the column super standard tableau of shape  $\lambda$ . Interpreting this statement in the coordinate ring  $A$ , this is stating that the product in  $A$  of the wedges

$$\bigwedge_{i \in C_1} v_i, \bigwedge_{i \in C_2} v_i, \bigwedge_{i \in C_3} v_i, \dots,$$

is equal to the product of the wedges

$$\bigwedge_{i \in C_1} u_i, \bigwedge_{i \in C_2} u_i, \bigwedge_{i \in C_3} v_i, \dots$$

However,  $A$  is a unique factorization domain [10, Section 9.2] and the equality of the products is exactly conditions (1), (2) and (3).  $\square$

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